Vector Fields

1. Sketch the vector field \( \mathbf{F}(x, y) = \frac{y \mathbf{i} - x \mathbf{j}}{\sqrt{x^2 + y^2}} \).

2. Sketch the vector field \( \mathbf{F}(x, y, z) = -y \mathbf{k} \).

3. Find the gradient vector field \( \nabla f \) of \( f(x, y) = \sqrt{x^2 + y^2} \) and sketch it.

4. Find the gradient vector field of \( f(x, y, z) = x \cos(y/z) \).

5. At time \( t = 1 \), a particle is located at position \( (1, 3) \). If it moves in a velocity field

\[
\mathbf{F}(x, y) = (xy - 2, y^2 - 10)
\]

find its approximate location at \( t = 1.05 \).

6. Determine whether \( \mathbf{F}(x, y) = e^x \cos y \mathbf{i} + e^x \sin y \mathbf{j} \) is a conservative vector field. If it is, find a function \( f \) such that \( \mathbf{F} = \nabla f \).

7. Determine whether \( \mathbf{F}(x, y) = (xy \cos xy + \sin xy) \mathbf{i} + (x^2 \cos xy) \mathbf{j} \) is a conservative vector field. If it is, find a potential function \( f \) for \( \mathbf{F} \).

8. (Hairy Ball Theorem)

   (a) Sketch a non-constant, non-vanishing, continuous, vector field on the unit square \([0, 1] \times [0, 1]\).

   (b) A torus can be obtained from a square by identifying (or “gluing”) the opposite sides with each other.

      i. Describe a non-constant continuous function \( f(x, y) \) such that \( f(x, 0) = f(x, 1) \)
          for all \( x \) and \( f(0, y) = f(1, y) \) for all \( y \). This function can now be viewed as a
          function on the torus.

      ii. Describe a non-constant, non-vanishing, continuous tangent vector field on the
          torus. [Hint: Consider \( \mathbf{F}(x, y) = f(x, y) \mathbf{i} + \mathbf{j} \), where \( f \) is the function above.

   (c) We now have a non-constant, non-vanishing, continuous vector field on the torus.
       The Hairy Ball Theorem asserts that there is no non-constant, non-vanishing, con-
       tinuous tangent vector field on a sphere. Imagine on the surface of the earth the
       wind velocity vector field. Interpret the Hairy Ball Theorem in the context of the
       wind velocity vector field.